Stochastic Calculus

Kas Wei

Contents

Chapter 1

Probabilities

1.1 Event Sets

<u>DEFINITION</u>. A collection (or call it a set) $\mathcal{F}$ of subsets of Ω is called a σ-algebra if it satisfies:

1. contains the empty set: $\emptyset \in \mathcal{F}$;
2. is closed under **countable** unions: $A_1, A_2, ..., \in \mathcal{F} \implies \cup_i A_i \in \mathcal{F}$;
3. is closed under complements: $A \in \mathcal{F} \implies A^C \in \mathcal{F}$;

It is trivial to know:

$$\begin{aligned} A_1, A_2, ... \in \mathcal{F} &\implies \cup_i A_i^C \in \mathcal{F} \\ &\implies (\cup_i A_i^C)^C \in \mathcal{F} \\ &\implies \cap_i A_i \in \mathcal{F} \end{aligned}$$

<u>TRIVAL σ-ALGEBRA</u>. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is a σ-algebra.
<u>COLLECTION OF ALL SUBSETS</u>. $\mathcal{F} = 2^\Omega$ is a σ-algebra.
<u>DEFINITION</u>. Let $\mathcal{G}$ be a collection of subsets of Ω. The σ-algebra generated by $\mathcal{G}$: $\sigma(\mathcal{G})$ is the **smallest** σ-algebra that contains $\mathcal{G}$.
<u>DEFINITION</u>. A pair is called a *measure space* $(\Omega, \mathcal{F})$ if Ω is the sample space and $\mathcal{F}$ is a σ-algebra of subsets.

1.2 Probability

<u>DEFINITION</u>. A **function** $\mathbb{P}$ defined on $(\Omega, \mathcal{F})$: $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a *probability measure* if:

1. $\mathbb{P}(\Omega) = 1$

2. Only for *countable* unions: if $A_i \cap A_j = \emptyset$ for $i \neq j \implies \mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$

The countable sample space is easy to handle with, take $\mathcal{F} = 2^\Omega$ and some $\mathbb{P}$ to assign each event to [0, 1]. But the uncountable space could be more delicate to solve.

1.3 Infinite Spaces

1.3.1 Uniform Lebesgue Measure on (0, 1)

Define a *Lebesgue measure* μ, then we can determine the probability that ω falls within an **open interval**:

$$\mathbb{P}(\{\omega : \omega \in (a, b)\}) = \mu((a, b)) := b - a, 0 < a \leq b < 1$$

In this case we notice that $2^\Omega = 2^{(0,1)}$ is not a σ-algebra of (0, 1). Then we introduce the *Borel σ-algebra* on (0, 1) to make an appropriate sample space for our experiment:

$$\mathcal{B}((0, 1)) := \sigma(\mathcal{O}) \text{ where } \mathcal{O} = \{A \subseteq (0, 1) : A = (a, b), 0 < a \leq b < 1\}$$

This note will not discuss much on the *Borel set*, it is about creating some subsets of open sets in Ω.

1.3.2 Infinite Sequence of Coin Tosses

Let $\omega = \omega_1\omega_2...\omega_i...\omega_n$ where $\omega_i \in \{H, T\}$. If I have known ω_1, ω_2, I can tell you if ω belongs to each of the sets in $\mathcal{F}_2$: all possible cases for two tossing. When n becomes very large, then we have:

$$\mathcal{F} = \sigma(\mathcal{F}_\infty),\ \mathcal{F}_\infty = \cup_\infty \mathcal{F}_n$$

But things (or call them sets) like "sequences for which x percent of coin tosses are heads" are not in $\mathcal{F}_\infty$, they are actually in $\mathcal{F}$.
Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, then if a set $A \in \mathcal{F}$ s.t. $\mathbb{P}(A) = 1$, then the event A occurs $\mathbb{P}$ **almost surely** (i.e. $\mathbb{P}$-a.s.).

Chapter 2

Information and Conditioning

2.1 Information and σ-Algebras

<u>Definition</u> 2.1.1. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a

Chapter 3

Brownian Motion

3.1 Scaled Random Walks

3.2 Brownian Motion

<u>DEFINITION</u>. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, a *Brownian motion* is a stochastic process $W = (W_t)_{t \geq 0}$ that:

1. $W_0 = 0$
2. independent increment: if $0 \leq r < s < t < u < \infty$ then $(W_u - W_t) \perp (W_s - W_r)$
3. stationary increment: if $0 \leq r < s$ then $W_s - W_r \sim \mathcal{N}(0, s - r)$
4. the map $t \to W_t$ is continuous for every ω

The distribution of a normally distributed random vector is uniquely determined by its **mean vector** and **covariance matrix**.
E.g. Let $W := (W_{t_1}, W_{t_2}, \ldots, W_{t_d})$ is a d-dimensional normally distributed variable, what is the covariance matrix? We have $\mathbb{E}[W_{t_1}, W_{t_2}, \ldots, W_{t_d}] = 0$. Then for $T \geq t$:

$$
\begin{aligned}
Co\mathbb{V}[W_T, W_t] &= \mathbb{E}W_T W_t - \mathbb{E}W_T \mathbb{E}W_t \\
&= \mathbb{E}W_T W_t \\
&= \mathbb{E}(W_T - W_t + W_t)W_t \\
&= \mathbb{E}[(W_T - W_t)W_t] + \mathbb{E}W_t^2 \\
&= \mathbb{E}[W_T - W_t]\mathbb{E}W_t + \mathbb{E}W_t^2 \\
&= \mathbb{V}W_t = t
\end{aligned}
$$

The covariance matrix is: $C = \begin{pmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_d \end{pmatrix}$

<u>Definition</u>. A *filtration* for the Brownian motion W is a collection of σ-algebra $\mathbb{F} = (\mathcal{F}_t)t \geq 0$:

1. Gathering more and more information: if $0 \leq s < t$ then $\mathcal{F}_s < \mathcal{F}_t$
2. Adaptivity: for all $t \geq 0$, we have $W_t \in \mathcal{F}_t$
3. Independence of future increments: if $u > t \geq 0$ then $(W_u - W_t) \perp \mathcal{F}_t$

By contrast, a Brownian motion W is a martingale under such filtration $\mathbb{F}$:

$$\mathbb{E}[W_t|\mathcal{F}_s] = \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] = 0 + W_s = W_s$$

3.3 Quadratic Variation

Let Π be a partition on $[0, T]$ and $||\Pi|| = \max_i(t_{i+1} - t_i)$.
<u>Definition</u>. The *first variation of f up to time T* is:

$$FV_T(f) := \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

<u>Lemma</u>. By the Mean Value Theorem there $\exists t_j^* \in [t_j, t_{j+1}]$ s.t.

$$f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$$
$$\implies FV_T(f) = \int_0^T |f'(t)| dt$$

<u>Definition</u>. Let $f : [0, T] \to \mathbb{R}$, the *quadratic variation of f up to time T* is:

$$[f, f]_T := \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

<u>Lemma</u>. $[W, f]_T = 0$ for any smooth function $f(t)$.

Proof. For any partition Π of $[0, T]$ we have:

$$\begin{aligned} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 &= \sum_{j=0}^{n-1} [f'(t_j^*)]^2 (t_{j+1} - t_j)^2 \\ &\leq ||\Pi|| \sum_{j=0}^{n-1} [f'(t_j^*)]^2 (t_{j+1} - t_j) \end{aligned}$$

Insert the inequality into the definition of $[f, f]_T$ to obtain:

$$\begin{aligned}[f, f]_T &\leq \lim_{||\Pi|| \to 0} ||\Pi|| \cdot \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f'(t_j^*)]^2 (t_{j+1} - t_j) \\ &= 0 \cdot \int_0^T |f'(t)|^2 dt = 0\end{aligned}$$

□

<u>Theorem</u>. Let W be a Brownian motion. Then for all $T \geq 0$ we have $[W, W]_T = T$ almost surely.

Proof. For a fixed partition $\Pi = \{0 = t_0, t_1, \ldots, t_n = T\}$ the sampled quadratic variation of W, denoted as Q_Π:

$$Q_\Pi := \textstyle\sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2$$

Since $Q_\Pi \to [W, W]_T$ as $||\Pi|| \to 0$, it is sufficient to show if $\mathbb{E}\Pi \to T$ and $\mathbb{V}\Pi \to 0$ then the theorem holds. By $W_{t_{j+1}} - W_{t_j} \sim \mathcal{N}(0, t_{j+1} - t_j)$:

$$\begin{aligned}\mathbb{E}[W_{t_{j+1}} - W_{t_j}]^2 &= t_{j+1} - t_j \\ \mathbb{V}[W_{t_{j+1}} - W_{t_j}]^2 &= 2(t_{j+1} - t_j)^2\end{aligned}$$

Take back to deliver that:

$$\begin{aligned}\mathbb{E}\Pi &= \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2 = \sum_{j=0}^{n-1} t_{j+1} - t_j = T \\ \mathbb{V}\Pi &= \sum_{j=0}^{n-1} (2(t_{j+1} - t_j)^2) \leq ||\Pi|| \sum_{j=0}^{n-1} 2(t_{j+1} - t_j) = 2\Pi T \to 0\end{aligned}$$

□

But it is incorrect to say that $dW_t^2 = dt$. In fact, $dW_t^2 \xrightarrow{d} dt$.
<u>Definition</u>. Let $f, g : [0, T] \to \mathbb{R}$. The covariation of f and g is:

$$[f, g]_T = \lim_{||\Pi|| \to 0} \textstyle\sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)][g(t_{j+1}) - g(t_j)]$$